Math 210C Lecture 23 Notes

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1 Characterizations of Semisimple Rings, Morita Equivalence, and Brauer Groups

1.1 Characterizations of semisimple rings

Last time, we were proving the following theorem.

Theorem 1.1. *The following are equivalent:*

1. *R is semisimple.*
2. *Every R-module is semisimple.*
3. *Every R-module is projective.*
4. *Every R-module is injective.*

Proof. We have proven (2) $\implies$ (1), (2) $\implies$ (3), and (3) $\iff$ (4).

(1) $\implies$ (2): R is a direct sum of simple module, and every simple module occurs in the sum. If $M \neq 0$ is an R-module and $0 \neq m \in M$, then consider the cyclic submodule $Rm \subseteq M$. R_m contain a minimal submodule (since R is artinian and Rm is a quotient of R, Rm is artinian). Then the collection X of semisimple submodules of M is nonempty. Given an chain C, the union $\bigcup_{NinC} N$ is semisimple. By Zorn's lemma, there is a maximal semisimple submodule N of M. Assume $N \neq M$. Then there is a simple submodule $A \subseteq M/N$. Let $B \subseteq M$ be the preimage of A under the quotient map. Let $b \in B \setminus N$. Then $Rb \to A$ is a surjection because A is simple. We claim that this is split. Given this, $B = N \oplus A$, so it is semisimple, contradicting maximality.

(4) $\implies$ (1): Let $I_1 \subseteq R$ be a simple left ideal (the following argument will show why we can find such an ideal). Then the injection $I_1 \to R$ is split, so $R = I_1 \oplus J_1$. Then the injection of a simple left ideal $I_2 \to J_1$ is split, so $J_1 = I_2 \oplus J_2$. Keep doing this to get $R = \varinjlim_n I_1 \oplus I_2 \oplus \cdots \oplus I_n \oplus J_n \cong \bigoplus_{j=1}^\infty I_j$. This is impossible, so the direct sum is finite. $\square$

1.2 Morita equivalence

For any ring, we have an equivalence of categories R-Mod $\to M_n(R)$-Mod. send $M \mapsto M^n$, the collection of column vectors. This is called a **Morita equivalence** of the rings R and $M_n(R)$. R^n is an $M_n(R)$-R bimodule (column vectors) P, and row vectors are an R-$M_n(R)$ bimodule of row vectors Q. Then $M \mapsto M^n \cong P \otimes_R M$ and an $M_n(R)$-module N gets sent to $N \mapsto Q \otimes_{M_n(R)} N$.

1.3 Brauer groups

Wedderburn's theorem implies that every finite dimensional central simple algebra over K is isomorphic to $M_n(D)$, where $Z(D) = K$.

Definition 1.1. Two central simple algebras A and B are **similar** if there exists a division algebra D with $Z(D) = K$ and $m, n \geq 1$ such that $A \cong M_n(D)$ and $B \cong M_m(D)$.

This gives an equivalence relation on central simple algebras. We can multiply similarity classes by $[A][B] := A \otimes_K B$. The identity is $[M_n(K)] = [K]$ because $M_n(K) \otimes_K B = M_n(B) \cong M_n(M_m(D))$. This multiplication is associative. The inverse of $[D]$ is $[D^{\mathrm{op}}]$: $D \otimes_K D^{\mathrm{op}} = M_n(K)$, where $n = \dim_K$. So this forms a group.

Definition 1.2. The group of similarity classes of CSAs over K is caller the **Brauer group** $\mathrm{Br}(K)$ of K.

A CSA $A = M_n(D)$ is split over an extension L of K if $A \otimes_K L \cong M_n(L)$.

Example 1.1. Let $\mathbb{H}$ be the quaterions, a 4 dimensional vector space over $\mathbb{R}$. Then $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$, where

$$i \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \qquad j \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$\mathbb{H}$ is split over $\mathbb{C}$, and $Z(\mathbb{H}) = \mathbb{R}$.

Remark 1.1. If $L/K \subseteq A$ is maximal, then A is split over L.

Let $f : G^2 \to L^\times$, where L/K is finite Galois, and $G = \mathrm{Gal}(L/K)$. Take the factor set (2 cocycle, $f(1, \sigma) = f(\sigma, 1) = 1$ for all $\sigma \in G$). Then $f \mapsto B_f$, a CSA over K containing L and of dimension $[L : K]^2$. The map f is used to define the multiplication. B_f has an L-basis consisting of elements $[\sigma][\tau] = f(\sigma, \tau)[\sigma\tau]$. The Galois action is $[\sigma] \cdot \alpha = \sigma(\alpha)[\sigma]$, where $\sigma \in G$ and $\alpha \in L$.

This give a map $H^2(G, L^\times) \to \mathrm{Br}(K)$. The image is $\mathrm{Br}(L/K)$, the collection of L-split classes in $\mathrm{Br}(K)$.

Example 1.2. $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ has order 2. $H^2(G, \mathbb{C}^\times) \cong R^\times / N_{\mathbb{C}/\mathbb{R}} \mathbb{C}^\times \cong \mathbb{Z}/2\mathbb{Z}$. We have that $H^2(G, \mathbb{C}^\times) \cong \mathrm{Br}(\mathbb{C}/\mathbb{R}) = \mathrm{Br}(\mathbb{R})$, so there are only 2 elements, $[\mathbb{R}]$ and $[\mathbb{H}]$.

1.4 Introduction to representation theory: representations and $F[G]$-modules

Let G be a group, and let F be a field. There is a 1 to 1 correspondence between $F[G]$-modules of F-dimension n and homomorphisms $G \to \mathrm{GL}_n(F)$. If V is an $F[G]$-module with $\dim_F(B)$, then we get $\rho : G \to \mathrm{Aut}_F(V)$ given by $g \mapsto (g : V \to V)$; $\mathrm{Aut}_F(V) \cong \mathrm{GL}_n(F)$ by a choice of basis. On the other hand, if we have $\rho : G \to \mathrm{GL}_n(F)$, then we have the $F[G]$-module $V = F^n$ with $g \cdot v = \rho(g) \cdot v$.